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Hierarchical *hp* Modeling and Locking Resolution in Laminated  
Plates and Shells (Grant F49620-95-1-0230)

Dr. Manil Suri  
Dr. Christoph Schwab

University of Maryland Baltimore County, Baltimore, MD 21250

Air Force Office of Scientific Research  
110 Duncan Avenue, Suite B115  
Bolling AFB, DC 20332

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(1) Non-conforming *hp* methods were developed for joining subdomains with non-matching boundary meshes on the interface. In particular, the mortar finite element method was shown to be optimal and stable. This work was done in conjunction with implementation in the commercial code MSC-NASTRAN. (2) *hp* finite element methods that are robust in the presence of locking and boundary layers were developed. These were applied to plates, shells, non-linear elasticity and non-Newtonian fluid flow. Both low-order and high-order locking-free mixed methods were investigated. (3) Analysis of *hp* space enrichment methods was carried out for non-linear contact problems. (4) An investigation into the finite element approximation of spectra of non-compact operators was initiated. The theory is needed for an *hp* approximation of solution to non-linear buckling problems.

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Newtonian flow

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## 2. Objectives

The following were the principal objectives of this project.

1. Modeling of multistructures via *hp* interface mixed methods for domain decomposition and concatenation. This work complemented an ongoing implementation in the commercial program MSC-NASTRAN.
2. Development of *hp* methods that are robust in the presence of locking and boundary layers. In particular, development of **mixed** *hp* methods. Applications to plates, shells, non-linear elasticity and non-Newtonian fluid flow.
3. Analysis of *hp* space enrichment methods and applications. The driving engineering application was one of non-linear contact problems.
4. Investigation of the finite element approximation of spectra of non-compact operators. Application in non-linear buckling analysis. (Work in progress.)

### 3. Summary of Accomplishments/Status of Effort

1. The search for a non-conforming  $hp$  method that can be used for domain decomposition and concatenation was **completed** by establishing that the  $hp$  mortar finite element method is essentially stable and optimal in terms of both  $h$  and  $p$  for a wide variety of meshes. These results were communicated to MSC-NASTRAN developers who incorporated a related  $hp$  method in their code. Remaining work (in progress) on this topic consists of analyzing some new, simpler versions of the method that we have formulated, including one that extends to three dimensions.
2. Research was **completed** on mixed  $hp$  finite element methods that were developed over straight-sided and curved quadrilaterals and triangles. This research will form the building blocks for future applications to various problems (e.g. viscoelasticity). Already, these methods were applied to a non-linear large displacement problem and shown to be more robust than the standard formulation. In a related project, stable  $hp$  mixed methods were developed for non-Newtonian fluid flow such as the upper convected Maxwell model.
3. The investigation of  $hp$  finite element methods for modeling boundary layers in thin domains was **completed**. In particular, rules for mesh-degree selection for the robust analysis of plates and shells with corners were developed, to take into account both the effect of corner singularities and boundary layers. Our prescriptions for handling boundary layers have been incorporated into e.g. the STRESS CHECK user's manual.
4. Motivated by an engineering solution method proposed by Barna Szabo for non-linear contact problems, an  $hp$  space enrichment method and its applications were investigated. Theorems that mathematically validated the method were established, setting the framework for future work in this area.
5. Research on low-order elements for plate and shell problems was continued with the completion of two papers that analyze the robustness of mixed FE methods in terms of both locking and boundary layers.
6. A method for the linearized analysis of non-linear buckling problems proposed by Barna Szabo (for STRESS CHECK) was investigated theoretically. Preliminary results on mathematical validity and finite element approximation were obtained. This investigation will form a significant topic in the research to be carried out under a subsequent AFOSR grant.

## 4. Accomplishments

### 1. Modeling of multistructures

We mathematically and computationally investigated non-conforming *hp* methods to model problems over a domain consisting of several subdomains. The motivations were to (1) distribute the discretization of a complicated domain among several users by breaking it into subdomains which can be recomposed for the final analysis, (2) allow selective refinement in regions where it is needed, by extracting the region, re-meshing it and putting it back, (3) discretize different, inter-connected PDEs over adjoining domains. Such methods also have applications in domain decomposition methods, e.g. the overlapping Schwarz algorithm.

One such method, involving three separate fields, was developed and tested by researchers at NASA Langley for **low-order** elements ( $p=1$  or  $2$ ). A version of this was under implementation by the MacNeal-Schwendler Corporation for their commercial program MSC-NASTRAN. Since MSC-NASTRAN uses high-order elements as well as low-order elements, their requirement was that the method used be such that it is *stable* and *optimal* in terms of all  $h$ ,  $p$  and  $hp$  methods that the user may choose (including meshes that are highly graded). The question then became one of developing a non-conforming method that satisfied all these criteria.

We fully solved this problem (for the two-dimensional case) under this grant, obtaining **sharp** theoretical and computational results. These were communicated to MacNeal-Schwendler and were used by them to select and fine-tune the final form of the method they implemented.

More specifically, we analyzed several *hp* non-conforming methods, and were able to show in [1], that the so-called "mortar finite element method" is an ideal choice. This is one of several methods that is based on the use of a Lagrange multiplier. If  $u_1$  and  $u_2$  are the test or trial functions used in different sub-domains  $\Omega_1$  and  $\Omega_2$ , then instead of the usual continuity condition

$$u_1 = u_2 \text{ on } \Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2,$$

the mortar FEM uses instead

$$\int_{\Gamma} (u_1 - u_2) \lambda \, ds = 0 \quad \forall \lambda \in S, \tag{1}$$

where  $S$  is a finite element space of Lagrange multipliers. Note that the meshes on  $\Gamma$  from the two domains  $\Omega_1$  and  $\Omega_2$  will not in general coincide.  $S$  is a space of piecewise polynomials on the mesh on  $\Gamma$  induced by the partition of one of the sides, say  $\Omega_1$ . If piecewise polynomials of degree  $p$  are used for  $u_1$  and  $u_2$ , then for  $S$  this method uses piecewise polynomials of degree  $p$  again, except that in the first and the last interval of the mesh, the degree is one less, i.e.  $p - 1$  (see [1] for details).

In the past, this method was only proposed for  $h$  refinement, and that too over meshes that were quasiuniform. Through theoretical analysis as well as computational testing, we established stability and optimality results for essentially arbitrary meshes, and with respect to  $h, p$  as well as  $hp$  refinement. In particular, exponential convergence was established for this method when geometric meshes with increasing  $p$  are used.

In this regard, the following theorems were proved for a model elliptic problem on a domain  $\Omega$  decomposed into subdomains  $\Omega_i$  [1].

**Theorem 1.** Let  $u$  be the exact solution, satisfying  $u \in H^l(\Omega)$ ,  $l > 3/2$  ( $l > 7/4$  if  $p$  varies), and  $u_{h,p}$  its  $hp$  mortar finite element approximation, using piecewise polynomials of degree  $p$  on quasiuniform meshes  $\{T_h^i\}$  on each  $\Omega_i$ . Then

$$\|u - u_{h,p}\|_{1,d} \leq Ch^{\mu-1}p^{-(l-1)+3/4}\|u\|_{l,\Omega}$$

where  $\mu = \min\{l, p + 1\}$  and  $C$  is a constant independent of  $h, p$  and  $u$ .

(Here,  $\|\cdot\|_{1,d}$  is the usual component-wise discrete  $H^1(\Omega)$  norm introduced by the non-conformity.)

**Theorem 2.** Let  $u_N$  denote the solution of the mortar  $hp$  method with  $N$  degrees of freedom, where geometric meshes are used so that the degree  $p$  is proportional to the number of layers  $n$  around each corner of  $\Omega$  in each sub-mesh. Then the error decreases exponentially, i.e. there exists a  $\gamma > 0$  such that

$$\|u - u_{h,p}\|_{1,d} \leq Ce^{-\gamma N^{1/3}}.$$

The paper [2] contains some additional results that indicate that the estimates obtained in [1] are sharp, and also that give computational results

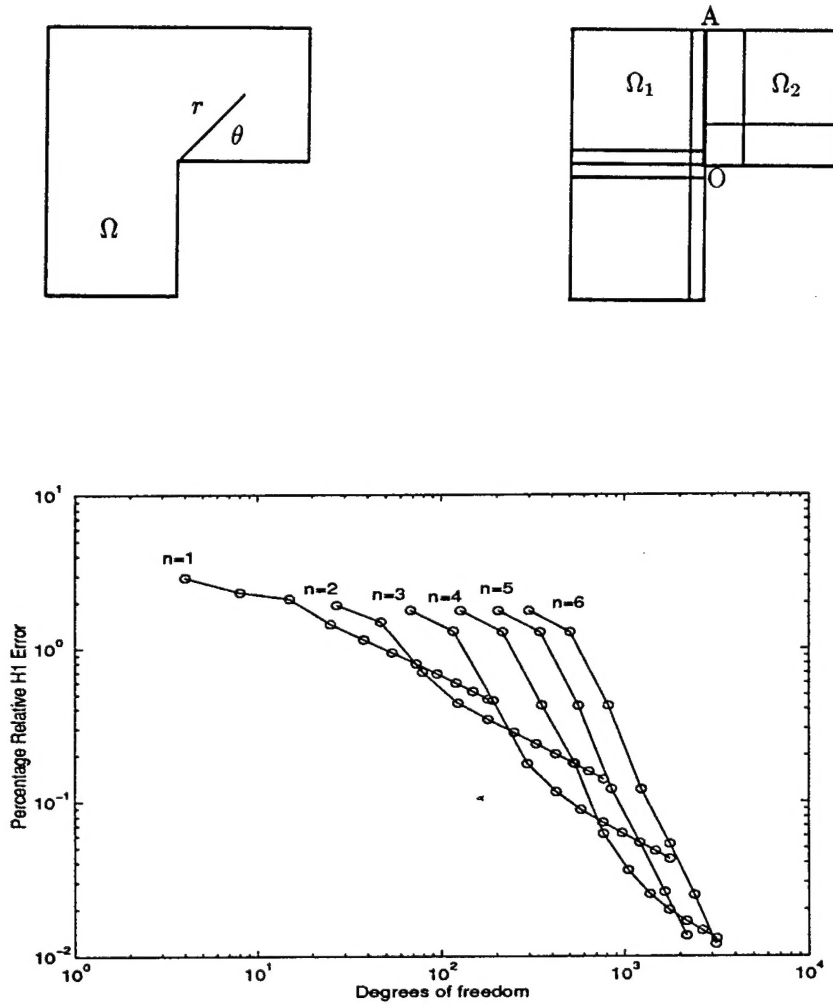


Figure 1: Partition of the L-shaped domain, and the relative error in the energy norm in dependence on  $N$  for geometric meshes

for a different test problem. We found that characteristic  $hp$  convergence curves are obtained (as predicted by Theorem 2 above), showing very little difference from the conforming method. This is illustrated in Figure 1, which shows the results for the mortar  $hp$  method over an L-shaped domain broken into two sub-domains. The mesh is geometrically refined over each domain, but with different geometric ratios, so that the meshes do not match on the interface.

These results will be further augmented in [3]. There, we have formulated a much simpler version of the mortar method, for which we can show the same numerical performance, and establish the same theoretical estimates as the method in [1]. The simplified method just uses degree  $p - 1$  uniformly over all intervals for the space  $S$ . (In fact, computational results indicate that the method keeps performing well even when degree  $p - 2$  is taken.)

The true use of this simplification will be seen in three dimensions. This is because in 3-d, the mortar finite element method is as yet not well-formulated except for the case of linear elements. Our idea of taking degree  $k - 1$ , however, generalizes easily to 3-d. The convergence properties again hinge on the stability of a certain projection operator  $\Pi$ , defined essentially by equation (1) (see [1]). Unlike the 2-d case where analytic tools were used to characterize the stability of  $\Pi$ , we will now have to rely on some computational tests, which we are in the process of developing. These results will once again be communicated to Dr. John Scheirmeier, our contact at MacNeal-Schwendler. This 3-d work will be the final section in the thesis [3], which has an expected completion date of July, 1998.

The impact of our results will be mainly via incorporation into MSC-NASTRAN, which is used in several DOD labs and aircraft manufacturing companies. There are several 3-d components (such as aircraft parts) for which good meshes have been pre-computed. This technology enables users to incorporate them in the final mesh, without having to worry about interelement continuity. Moreover, selective refinement of the mesh or the degree can be implemented, where required.

## 2. Development of mixed $hp$ finite elements

One of our ongoing goals in this project has been to develop and analyze a library of mixed  $p$  and  $hp$  finite element methods. Such methods have been very useful in the traditional  $h$  version context, where they have been used



in problems such as elasticity, plates and shells, where they overcome the effects of locking. They have also proved essential in other applications (e.g. Stokes flow, non-Newtonian flow, plasticity, viscoelasticity and viscoplasticity), where the variational formulation involves a saddle point, necessitating a mixed method. It is anticipated that with the increasing emphasis on  $p$  and  $hp$  methods, and the application of these methods to various new problems, a library of validated mixed  $hp$  elements will be valuable for Air Force needs. More specifically, such elements are essential when problems involving the constraint of incompressibility are to be solved, such as e.g. the deformation of rubber tyres. Future applications of this work will include e.g. the failure of electronic components, for which the soldering material can display viscoelastic (incompressible) behavior at high temperatures.

In previous work, we developed and analyzed mixed  $hp$  methods on parallelograms, and showed that with these, the stresses can be accurately recovered, for any Poisson ratio (note that the  $p$  version is *not* free of locking for the stresses). Under this grant, we computationally tested these elements and also extended our work to triangles and to curved elements. This was the Ph.D. dissertation [4] by our student, Lt. Col. Lawrence Chilton. Among other results, we (1) developed a curved quadrilateral element for which we can theoretically establish the same stability as the straight-sided parallelogram element, (2) developed mixed  $hp$  triangular elements which are stable in  $h$  and demonstrate an inf-sup constant of  $O(p^{-1})$  in the range of  $p$  used in practice, (3) demonstrated the accuracy of these elements through computational experiments.

Let us describe some of these results from [4] in more detail. First, for parallelograms, we considered various combinations, the most well-known of which are the following. (Since these mixed elements are used for the elasticity equations as well as Stokes' problem, we describe them for the latter, in which case the two unknowns are the velocity and the pressure, instead of the displacement and the sum of the normal stresses.)

(A)  $[Q_p]^2/Q_{p-2}$

Here, the set of degrees of freedom used on each parallelogram element is  $Q_p$  for each component of the velocity, and  $Q_{p-2}$  for the pressure. This element is popular especially in spectral element methods. The combination is stable in terms of  $h$ , while the stability behaves like  $O(p^{-1/2})$  in terms of the degree. In terms of  $p$ , the element gives optimal convergence in the ve-

locity (or displacement) and no worse than a loss of  $O(p^{1/2})$  in the pressure (or stresses). However, it does not give the optimal approximation of the velocity in  $h$ , since the pressure degree is not balanced properly according to approximation theory. Hence, it is not recommended for  $hp$  FEM.

(B)  $[Q_p]^2/P_{p-1}$

This combination avoids the pitfalls of the above element, by being optimal (and stable) in  $h$ . Recently, it was shown by Bernardi and Maday that the stability holds in  $p$  as well, with **no** loss, so that this is an ideal  $hp$  element.

(C)  $[Q_p \cap P_{p+2}]^2/P_{p-1}$

This element has fewer degrees of freedom than (B). Although this reduction appears to reduce the theoretical stability estimate to  $O(p^{-1/2})$ , our computational results in [4,5] indicate that it behaves as well as (A).

Let us now describe the corresponding methods for curved elements, which are of particular interest in the  $p$  version, where curved boundaries must be modeled **exactly** or with sufficient accuracy. Consider a single curved element  $K$  in the mesh, where  $K$  is the image of the reference square  $\hat{K} = [-1, 1]^2$  under a smooth, invertible mapping  $F_K$ . Let  $V(\hat{K})$  be the velocity space on the reference element (e.g. for method (A),  $V(\hat{K}) = [Q_p(\hat{K})]^2$ ). The usual way to define the velocity space is to take

$$V(K) = \{v | v = \hat{v} \circ F_K^{-1}, \hat{v} \in V(\hat{K})\},$$

with the pressure space being defined similarly.

Unfortunately, this definition does not yield a combination for which an inf-sup condition can be easily proven, raising questions about its stability. Therefore, in [4], we came up with an alternative definition. We divided the basis of  $V(\hat{K})$  into two sets: the *internal* (bubble) basis functions which vanish on  $\partial\hat{K}$  and the *external* basis functions  $E(\hat{K})$  which are non-zero on part of  $\partial\hat{K}$ . Then we defined

$$\begin{aligned} I(K) &= \{v | v = \mathcal{P}_K(\hat{v}), \hat{v} \in I(\hat{K})\} \\ E(K) &= \{v | v = \hat{v} \circ F_K^{-1}, \hat{v} \in E(\hat{K})\}, \end{aligned}$$

where  $\mathcal{P}_K$  represents the **Piola transform** from  $\hat{K}$  onto  $K$ . We took  $V(K)$  to be the space spanned by  $\{I(K), E(K)\}$ , and defined the pressure as before.

With this definition, the same stability is obtained as that for straight-sided elements. Computational results confirmed that the method is optimal in **both**  $h$  and  $p$ .

Finally, for triangular elements, we took the combination  $[P_p]^2/P_{p-2}$ . Extensive numerical investigation in [4] indicated that the stability behaves like  $O(p^{-1})$  for this combination. In [7], we also theoretically analyzed this element, where we showed that it is stable with respect to  $h$ , while the  $p$  stability is no worse than  $O(p^{-3})$ . We also showed exponential convergence there for fluid flow problems.

Several numerical experiments are given in [4]. One of them concerns a benchmark problem suggested by Szabo and Babuška, for the stress extraction around a circular hole. Figure 2 shows this test problem, and also shows a comparison of the FE solutions obtained by the standard  $p$  version FEM (such as via STRESS CHECK), and one using our mixed  $p$  version FEM. (We are calculating SNS, the sum of the normal stresses here.)

The results in [4] were also extended to non-linear problems with large displacement and small strain, when the Poisson ratio is close to 0.5 (e.g. rubber). We showed that our elements work better than the standard  $hp$  finite element method. (For instance, for Poisson ratio 0.499, the non-linear solver in STRESS CHECK fails to calculate accurate stresses, unlike our method.) Our future goal is to investigate the use of these mixed methods in various applications, such as non-Newtonian fluid flow (see below) and viscoelasticity.

### 3. Mixed $hp$ methods for non-Newtonian flow

We have applied the results from [4,5,6] to a model problem in CFD, the upper convected Maxwell fluid (which models viscoelastic flow). This is generally solved using a “three field” mixed formulation (velocity, pressure and stress). A question of crucial importance that then occurs is how to balance these spaces to ensure stability. The usual method that is used for analysis is to consider the limiting case of zero relaxation time, and formulate spaces that are stable for this (linearized) case. Although this method has been used for  $h$  version spaces, it has not so far been used to design  $p$  or  $hp$  spaces. In fact, the computational study of such flows by the  $p/hp$  methods has only recently been initiated, by Crochet and Khomami.

In [7], we derived various choices of  $hp$  spaces for this problem, and char-

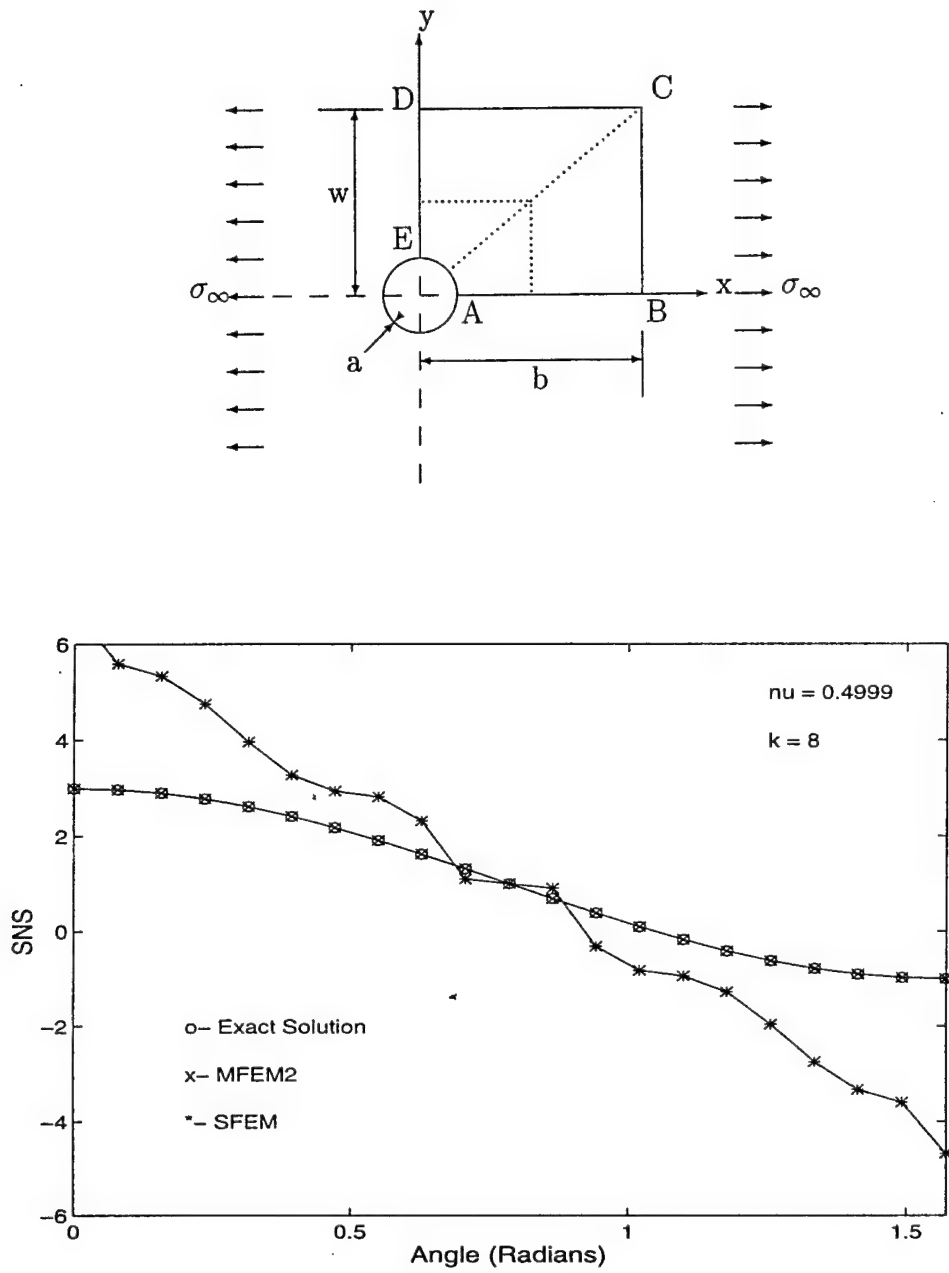


Figure 2: *Top*: Rigid circular inclusion benchmark problem and mesh. *Bottom*: The standard (SFEM) and mixed (MFEM) finite element methods under  $p$  refinement.  $\nu = .4999$ . SNS along circular boundary. Degree=8.

acterized their stability and convergence properties. In particular, we (1) showed that the choice of using  $Q^{p+1}$  elements for the (discontinuous) stresses,  $P^p$  for the (discontinuous) pressures and  $Q^{p+1}$  for the (continuous) velocities is completely **optimal** and **stable** in terms of **both**  $h$  and  $p$ , and (2) established exponential  $hp$  convergence for several choices (all results being for the limiting case of zero relaxation time). We also analyzed the  $hp$  modified EVSS method, which allows us to take smaller spaces for the stresses (in fact, for the optimal choice above, we can now take only the reduced space  $Q^{p'}$  for the stresses).

In addition to formulating these new combinations of spaces, our work also provided a mathematical validation of spaces similar to those used by Khomami. Here, **continuous** stresses are generally used, and we obtained estimates for the following combinations:

For **quadrilaterals**:  $Q^{p+3}$  stresses,  $P^p$  or  $Q^{p-1}$  pressures and  $Q^{p+1}$  velocities.  
For **triangles**:  $P^{p+3}$  stresses,  $P^{p-1}$  pressures and  $P^{p+1}$  velocities.

Finally, we showed that for graded geometric meshes, we obtain exponential  $hp$  convergence computationally for a stationary Newtonian flow in an L-shaped domain. This models the singularities arising also in non-Newtonian flow at the reentrant corners in the so-called 4:1 contraction problem (see Figure 3).

#### 4. Linearized analysis of non-linear buckling

The area of failure prediction, in the context of e.g. electronic components, ceramics, laminated composites, etc, has been of long-standing importance to USAF needs. Experimental determination of loads that can cause buckling and other failures is both expensive and unreliable. Mathematical models used in current engineering practice are, on the other hand, based on dimensionally reduced descriptions of an elastic body. The assumptions necessary for such dimensional reduction to hold can often result in a large, unknown modeling error when non-zero initial stress states are present, leading to results such as unnecessary weight penalties and unexpected failures.

We have been mathematically validating a method proposed by Barna Szabo that does buckling analysis for the fully three-dimensional problem at hand, rather than some asymptotic (dimensionally reduced) limit. This method determines buckling loads and deformations accurately and inexpensively, using  $hp$  methods to obtain high rates of convergence even in the

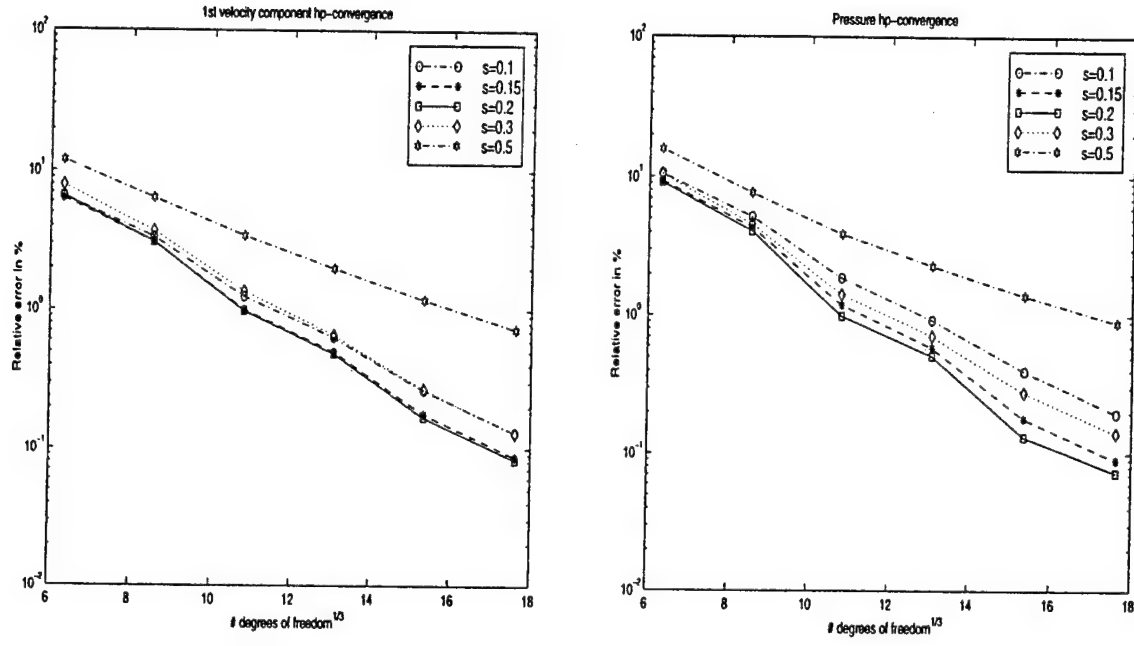


Figure 3:  $H^1(\Omega)$  error in velocity and  $L_2(\Omega)$  error in pressure versus  $N^{1/3}$  for various grading factors  $s$ .

presence of singularities.

The formulation by Szabo reduces to the question of finding the spectral values of a *non-compact* operator. Such problems can be quite difficult to approximate numerically, due to the presence of continuous and residual spectra, and due to spurious eigenvalue approximations. The first question that must be addressed for a non-compact operator is whether the continuous or residual spectrum exist, or whether things are better behaved, and one has only eigenvalues (note that approximate methods generally are not successful if non-empty continuous or residual spectra need to be calculated).

Under this grant, we established that in regions of ellipticity (which are the regions of interest), the spectrum only consists of real, isolated eigenvalues of finite multiplicity. This clears the way for approximation by numerical methods. Moreover, we proved that for every eigenvalue  $\lambda$ , there exists a sequence of approximate eigenvalues converging to  $\lambda$  when the finite element method is used.

We are currently dealing with the question of spurious eigenvalues. For certain situations, we can show that these spurious modes will not pollute the lowest eigenvalue (which is the one of interest). For other situations, we have developed an algorithm that can detect (and hence eliminate) such spurious modes. Ongoing research will concentrate on questions such as the treatment of non-conservative loads, as well as the question of how to deal with singular domains. Upon completion of this research, we will have both a robust numerical method suitable for implementation in the commercial *hp* code STRESS CHECK, and a mathematically rigorous formulation of buckling of three-dimensional bodies, which incorporates classical engineering derivations.

## 5. *p* version space enrichment methods

Space enrichment methods have traditionally been used in the context of the *h* version. In a recent paper by Szabo and Volpert, a *p* version space enrichment method was shown to be superior to the usual *hp* version for contact problems. These (non-linear) problems occur in a variety of situations (e.g. gears and machinery of every kind), and are related to wear and failure of components.

We completed the collaboration [8] with Costabel and Dauge (University of Rennes, France). In our paper, we obtained a theoretical justification of

the  $p$  version space enrichment method in one dimension, which shows why the 2-d formulation by Szabo and Volpert works so well. We also derived an integral equation formulation which can be used with space enrichment to give robust exponential convergence when the singularities lie in a known range. Computational results were also provided.

## 6. The $hp$ approximation of boundary layers.

Given the  $hp$  capability available through several  $hp$  codes, we showed how the meshes and degrees should be designed when boundary layers are present. This formed the Ph.D. thesis of our student, Christos Xenophontos [9], which built upon the one-dimensional results obtained in [10]. The problem Xenophontos investigated was

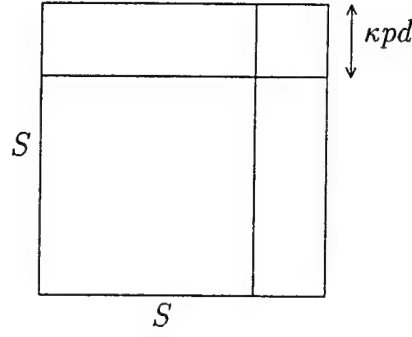
$$-d^2 \Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

for which he proved exponential convergence in  $d$ . He looked both at the case of a smooth domain and the case of a square (unsmooth) domain (these results appearing, respectively, in [11],[12]). In addition, the application of our methods to plate and shell problems was also carried out [13], since these two-dimensional models have boundary layers near the edges as well. A key aspect here was to choose the meshes and the degrees such that *both* boundary layers and corner singularities are optimally approximated.

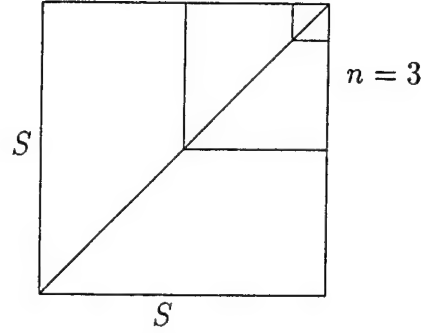
Let us illustrate the main idea for a square. In Figure 4, we have shown four possible meshes for a quarter of a square plate, of thickness  $d$ . It is assumed that polynomials of degree  $p$  are being used ( $p$  version), while  $\kappa$  is a constant. The first mesh shows the refinement needed to treat just the boundary layer. The second is the mesh that would be used just to treat the corner singularity. The third superimposes the first two meshes. The fourth is the mesh that really should be used, taking into account the nature of the singularities at the corner which essentially behave like  $(\frac{r}{d})^\alpha$ . Numerical experiments in [9,13] indicate that whereas the first mesh is adequate for getting a good error in the energy norm, it is the fourth mesh that is necessary if point-wise convergence in the stresses is desired.

Our results were incorporated into the user guidelines for STRESSCHECK (see the STRESSCHECK manual and the technical brief by C. Schwab from ESRD).

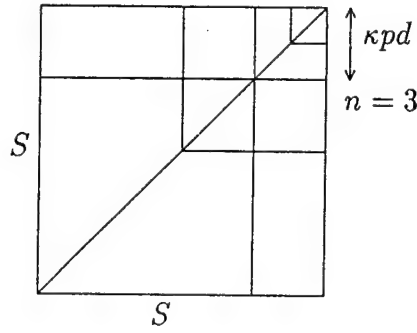




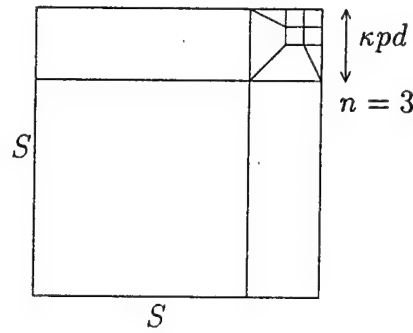
Scheme 1: BL-mesh



Scheme 2: Corner mesh



Scheme 3: BL-corner mesh



Scheme 4: BL-refined corner mesh

Figure 4: Meshes on a square. *Scheme 1*: Boundary layer mesh. *Scheme 2*: Geometric refinement for  $d = 1$ , with  $n = 3$  layers. *Scheme 3*: Union of *Scheme 1* and *Scheme 2*. *Scheme 4*: Union of *Scheme 1* and  $d$ -dependent geometric refinement with  $n = 3$  layers.

## 7. Locking-free finite elements for plates and shells

*Locking* is the phenomenon by which the numerical approximation of parameter-dependent problems deteriorates for values of the parameter close to a limiting value.

Joint work with J. Pitkäranta [14] was motivated by the locking constraints found in shell problems. So far, satisfactory mixed method techniques to handle such constraints have not been developed. A natural strategy would be to extend the derivation of plate elements to the shell setting. However, mixed methods for the Reissner-Mindlin plate are derived by appealing to a Helmholtz decomposition, an indirect technique without an apparent analog for shells.

In our work, we developed a new means of deriving mixed elements for the Reissner-Mindlin plate, which *does* extend to shells. For the first time, the reduction operators and polynomial spaces used to treat the constraints are *directly* derived. Our analysis gives minimal conditions on plate elements needed to prevent locking, and also to approximate the boundary layers present. Using our analysis, a clear picture emerges for the comparison of currently available low-order ( $h$  version) plate elements, such as the MITC and Arnold-Falk ones.

In [15], we further extended this work, by comparing three low-order plate elements in their ability to deal not only with locking, but also boundary layers. We found that the MITC elements are more robust in this regard than the so-called Arnold-Falk or Arnold-Brezzi elements. We also showed how these elements could be modified at the boundary to regain robustness. These results are now being used to analyze shell problems, where the interaction between locking and boundary layers is more complicated.

## 8. Work completed by C. Schwab

Christoph Schwab was supported under this grant for the first year only, after which he took up a position at ETH Zurich. In addition to collaborating on projects 3 and 6 above, let us mention two other projects that he completed while on this grant.

1. Wavelet Galerkin Discretization of Boundary Integral Equations on Surfaces

In joint work with T. von Petersdorff and R. Schneider [16], a fully discrete Multiscale (wavelet) method for boundary integral equations on surfaces in  $R^3$  was analyzed. It was shown that sparse stiffness matrices with  $O(N(\log N)^2)$  nonzero entries suffice to preserve the full asymptotic convergence rate of the scheme. An explicit quadrature strategy was presented, which allows the computation of these entries with the necessary accuracy in  $O(N(\log N)^4)$  steps (here,  $N$  denotes the number of degrees of freedom on the boundary surface). These results are true for a wide class of integral equation formulations stemming from, e.g., potential theory, elasticity, exterior Stokes flow, Maxwell's equations and Helmholtz equations (moderate frequencies).

## 2. Hierarchical modeling in mechanics

A project on hierarchical modeling for various problems in mechanics was completed, under partial sponsorship of AFOSR. An important aspect of this work was to develop estimators that could be used to predict elements where the model order needed to be increased. The review article [17] was written for a summer school in Leicester, England, and forms one of six chapters in the proceedings.

## 5. Participating Personnel

In addition to the PIs Manil Suri and Christoph Schwab, the following people were associated with the research:

Collaborators: Barna Szabo (University of Washington, St. Louis, MO), Ivo Babuška (University of Texas, Austin, TX), Monique Dauge and Martin Costabel (University of Rennes, Rennes, France), Juhani Pitkaranta (Helsinki University of Technology, Helsinki, Finland)

Ph.D. students: Christos Xenophontos, Lt. Col. Lawrence Chilton, USAF and Padmanabhan Seshaiyar.

Summer salary support was provided for P. Seshaiyer through the current grant.

## 6. Theses and journal publications

NOTE: Most publications, including recent preprints, can be retrieved from M. Suri's web page, <http://www.math.umbc.edu/~suri>.

1. P. Seshaiyer and M. Suri, "Uniform  $hp$  convergence results for the mortar finite element method," (1997 preprint, to appear in *Mathematics of Computation*, 1999).
2. P. Seshaiyer and M. Suri, "Convergence results for non-conforming  $hp$  methods: The mortar finite element method," *Contemporary Mathematics*, 218, 467-473 (1998).
3. P. Seshaiyer, "Non-conforming  $hp$  finite element methods," Ph.D. Dissertation, University of Maryland Baltimore County, expected July 1998.
4. L. Chilton, "Locking-free mixed  $hp$  finite element methods for linear and geometrically nonlinear elasticity," Ph.D. Dissertation, University of Maryland Baltimore County, May, 1997.
5. L. Chilton and M. Suri, "On the selection of a locking-free  $hp$  element for elasticity problems," *Int. J. Numer. Meth. Eng.* 40, 2045-2062 (1997).
6. L. Chilton and M. Suri, "Stable and optimal mixed  $hp$  methods over curvilinear domains," (in preparation).
7. C. Schwab and M. Suri, "Mixed  $hp$  finite element methods for problems in non-Newtonian and Stokes flows," (1997 preprint, to appear in a special issue of *Comp. Meth. Appl. Mech. Eng.* devoted to high-order CFD methods, 1999).
8. M. Costabel, M. Dauge and M. Suri, "Numerical approximation of a singularly perturbed contact problem," (1996 preprint, to appear in *Comp. Meth. Appl. Mech. Eng.*, 1998).
9. C. Xenophontos, "The  $hp$  version of the finite element method for singularly perturbed problems," Ph.D. Dissertation, University of Maryland Baltimore County, April, 1996. (thesis can be retrieved from the web address <http://www.clarkson.edu/~christos/thesis.ps>)
10. C. Schwab and M. Suri, "The  $p$  and  $hp$  versions of the finite element method for problems with boundary layers," *Mathematics of Computation*, 65, 1403-1429 (1996).

11. C. Xenophontos, "The  $hp$  finite element method for singularly perturbed problems in smooth domains," *Math Models and Methods in Applied Sciences*, Vol 2, No. 8 , 1998.
12. C. Xenophontos, "The  $hp$  finite element method for singularly perturbed problems in non-smooth domains," (1997 preprint, submitted to *Numerical Methods for PDEs*).
13. C. Schwab, M. Suri and C. Xenophontos, "The  $hp$  finite element method for problems in mechanics with boundary layers," (1996 preprint, to appear in *Comp. Meth. Appl. Mech. Eng*, 1998).
14. J. Pitkäranta and M. Suri, "Design principles and error analysis for reduced-shear plate-bending finite elements," *Numerische Mathematik*, 75, 223-266 (1996).
15. J. Pitkäranta and M. Suri, "Upper and lower plate-bending error bounds for some plate-bending finite elements," (1998 preprint, submitted to *Numerische Mathematik*, 1998).
16. T. von Petersdorff, C. Schwab and R. Schneider, "Multiwavelets for second kind integral equations," *SIAM J. Numer. Anal.*, 34, 2212-2227 (1997).
17. C. Schwab, "Hierarchic Modeling in Mechanics," (1996 preprint, to appear as a survey chapter in the Proceedings of the 1996 Leicester Summer School in Numerical Mathematics, Springer-Verlag).

## 7. Interactions/Transitions

### Presentations at meetings, conferences and seminars.

- Mar '95** Texas A & M University, College Station, TX, "The  $hp$  finite element modeling of plates and shells" (seminar)
- Mar '95** University of Texas, Austin, TX, "The  $hp$  finite element modeling of plates and shells" (seminar)
- Jul '95** ICIAM '95 (International Congress of Industrial and Applied Mathematics), Hamburg, July 3-7, 1995, "Mixed  $hp$  methods for plates and shells" (Minisymposium lecture, organized minisymposium)

- Aug '95** "Waves and Memory in Continua: A Meeting in Honor of Richard C. MacCamy," Carnegie-Mellon University, Pittsburgh, Aug 17-19, "The finite element modeling of thin structures" (invited plenary lecture)
- Oct '95** "1995 SIAM Annual Meeting," Charlotte, North Carolina, Oct 23-26, "The design of locking-free reduced-shear plate elements" (invited minisymposium lecture)
- Dec '95** Courant Institute, New York University, New York, NY, "The *hp* finite element modeling of thin structures" (seminar)
- Jan '96** Washington University, St. Louis, MO, "The *hp* finite element modeling of thin domains" (seminar)
- Mar '96** "Engineering Problems: Mathematical Formulation, Analysis, and their Computational Treatment," University of Maryland College Park, Mar 21-24, "The *hp* finite element modeling of thin structures" (invited plenary lecture)
- Jul '96** "Prague Mathematical Conference 1996," Prague, Czech Republic, Jul 8-12, "The *hp* finite element modeling of thin structures" (invited plenary lecture)
- Jul '96** "Numerical Methods and Computational Mechanics in Science and Engineering," Miskolc, Hungary, Jul 15-19, "The *hp* approximation of boundary layers" (invited minisymposium lecture)
- Aug '96** Ecole Polytechnique Federale de Lausanne, Lausanne, Switzerland, "The *hp* finite element modeling of thin domains" (seminar)
- Sep '96** University of Maryland College Park, College Park, MD, "The design of locking-free reduced-shear plate elements" (seminar)
- Oct '96** University of Texas, Austin, TX, "The design of locking-free reduced-shear plate elements" (seminar)
- Nov '96** University of Colorado, Denver, CO, "The *hp* finite element modeling of thin domains" (seminar)

- Apr '97** American Mathematical Society Meeting, College Park, MD, April 12-13, "Uniform  $hp$  estimates over partitioned domains" (invited minisymposium lecture)
- Apr, '97** Semi-annual FEM conference, New York University, New York, NY, "Eigenvalue analysis for a three-dimensional buckling model" (contributed lecture)
- Jun '97** "First International Congress of the International Society for Analysis, Applications and Computing," University of Delaware, Newark, DE, Jun 2-6, "Mixed  $hp$  methods for elasticity and flow problems" (invited minisymposium lecture)
- Aug '97** "US Congress on Computational Mechanics," San Francisco, CA, Aug 6-8, "Eigenvalue analysis of a three-dimensional buckling model" (invited minisymposium lecture)
- Aug '97** "Tenth International Conference on Domain Decomposition," Boulder, CO, Aug 10-14, "Uniform  $hp$  estimates for partitioned domains" (contributed minisymposium lecture)
- Sep '97** Pennsylvania State University, PA, "Uniform  $hp$  estimates over partitioned domains" (seminar)
- Oct, '97** Semi-annual FEM conference, Cornell University, Ithaca, NY, October 10-11, 1997, "The  $hp$  version for non-Newtonian fluid flow problems" (contributed lecture)

## Transitions

1. Work on non-conforming  $hp$  methods for multistructures validated and helped develop an interface method which was incorporated in the commercial code MSC-NASTRAN (a product of the MacNeal-Schwendler Corporation). Our results were used in designing the final method implemented. Contact: Dr. John Schiermeier, The MacNeal-Schwendler Corporation, 815 Colorado Blvd, Los Angeles, CA 90041-1777 (phone: 213-259-3832)
2. Research on non-linear buckling was initiated in response to questions posed by Prof. Barna Szabo of ESRD, 7750 Clayton Road, St. Louis, Missouri 63117 (phone: 314-645-1423). This work is geared towards formulating

an implementation on the commercial code STRESS CHECK. Our work on formulating guidelines for  $hp$  meshes in the presence of boundary layers has already been used in this code.

**8. Inventions/patent disclosures:** None.

**9. Honors and Awards:** None